# COMPACT KÄHLER MANIFOLDS WITH NONPOSITIVE BISECTIONAL CURVATURE

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Abstract. For a compact Kähler manifold with nonpositive bisectional curvature, we show that a finite cover is biholomorphic and isometric to a flat torus bundle over a compact Kähler manifold with $c_1 < 0$. This confirms a conjecture of Yau.

## 1. Introduction

The uniformization theorem of Riemann surfaces says the sign of curvature could determine the conformal structure in some sense. Explicitly, if the curvature is positive, it is covered by either $\mathbb{P}^1$ or $\mathbb{C}$. On the other hand, if the curvature is less than a negative constant, it is covered by the unit disk $\mathbb{D}^2$.

It is natural to wonder whether there are generalizations in higher dimensions. For the compact case, the famous Frankel conjecture says if a Kähler manifold has positive holomorphic bisectional curvature, then it is biholomorphic to $\mathbb{CP}^n$. This conjecture was solved by Mori [8] and Siu-Yau [9] independently. In fact Mori proved the stronger Hartshorne conjecture. Later, Mok [7] solved the generalized Frankel conjecture, the result says that, if a compact Kähler manifold has nonnegative holomorphic bisectional curvature, then the universal cover is isometric-biholomorphic to $(\mathbb{C}^k, g_0) \times (\mathbb{P}^{n_1}, \theta_1) \times \cdots \times (\mathbb{P}^{n_l}, \theta_l) \times (M_1, g_1) \times \cdots \times (M_i, g_i)$, where $g_0$ is flat; $\theta_k$ are metrics on $\mathbb{P}^{n_k}$ with nonnegative holomorphic bisectional curvature; $(M_j, g_j)$ are compact irreducible Hermitian symmetric spaces.

If the curvature is negative, the current knowledge is much less satisfactory. For example, a famous conjecture of Yau says if a simply connected complete Kähler manifold has sectional curvature between two negative constants, then it is a bounded domain. So far, it is not even known whether there exists a nontrivial bounded holomorphic function on such manifolds.

As in the Riemannian case, it is often important to understand the difference between the negative curved case and nonpositive case. The former tends to be hyperbolic in some sense, while the latter usually possesses some rigidity properties. For compact Kähler manifolds with nonpositive bisectional curvature, there is a conjecture of Yau:

**Conjecture.** *Let $M^n$ be a compact Kähler manifold with nonpositive bisectional curvature. Then there exists a finite cover $M'$ of $M$ such that $M'$ is a holomorphic and metric fibre bundle over a compact Kähler manifold $N$ with nonpositive bisectional curvature and $c_1(N) < 0$, and the fibre is a flat complex torus.*

In [11], Wu and Zheng proved this conjecture under the assumption that the metric is real analytic. Their proof could be divided into two steps: 1. At the points where the Ricci curvature has maximal rank, they proved that the foliation of the kernel of the curvature tensor is a holomorphic foliation. 2. The leaves of the foliation close up.

In this note we confirm this conjecture in full generality.

**Theorem 1.** *Let $(M^n, g)$ be a compact Kähler manifold with nonpositive bisectional curvature. Then there exists a finite cover $M'$ of $M$ such that $M'$ is a holomorphic and metric fibre bundle over a compact Kähler manifold $N$ with nonpositive bisectional curvature and $c_1(N) < 0$, and the fibre is a flat complex torus.*

The proof of theorem 1 uses Hamilton's Ricci flow [5] and Hamilton's maximum principle for tensors([6][2][1]), together with some argument in [11] by Wu and Zheng. The construction of the invariant convex set is analogous to that in [1] by Böhm and Wilking. The key point is to prove that there exists a small $\epsilon > 0$ such that the after the Ricci flow, $Ric(g_t) \leq 0$ for all $0 < t < \epsilon$(note that the bisectional curvature is not necessarily nonpositive for small $t$).

**Remark.** *There is a general philosophy that the Ricci flow makes the curvature towards positive. So it might be interesting to see that in our case, at least in a short time, the Ricci curvature remains nonpositive.*

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## 2. The proof of theorem 1

*Proof.* Let $g(t)$ be the solution to the Ricci flow equation $\frac{\partial g(t)}{\partial t} = -2Ric(g(t))$ with $g(0) = g$. We shall construct a family of convex cones $V_t$ which are invariant under parallel transport such that the curvature tensor of $g(t)$ lies inside $V_t$ for small $t$.

**Proposition 1.** *Let $V_t$ be a family of Kähler algebraic curvature operators satisfying the following conditions:*

*(1). $Ric(\alpha, \overline{\alpha}) \leq 0$ for any $\alpha \in T^{1,0}M$.*

*(2). $|R_{x\overline{x}u\overline{v}}|^2 \leq (1 + tK_1)Ric(u, \overline{u})Ric(v, \overline{v})$ for any $x, u, v \in T^{1,0}M$ and $|x|_{g(t)} = 1$.*

*(3).$\|R\| \leq K_2 + tK_3$.*

*Then for suitable positive constants $K_1, K_2, K_3$, there exists a $\epsilon > 0$ such that the $V_t$ is invariant under the Ricci flow for $0 \leq t < \epsilon$.*

*Proof.* First, we prove $V_t$ is a convex for each $t$. It is easy to see that condition (1) and (3) defines a convex set. For condition (2), suppose $R, S$ are two tensors satisfying (1), (2), then for any $0 \leq \lambda \leq 1$, define

$$T = \lambda R + (1 - \lambda)S.$$

(1)
$$
\begin{aligned}
|T_{x\bar{x}u\bar{v}}|^2 &= |\lambda R_{x\bar{x}u\bar{v}} + (1-\lambda)S_{x\bar{x}u\bar{v}}|^2 \\
&\leq (1+tK_1)|\lambda\sqrt{Ric_R(u,\overline{u})Ric_R(v,\overline{v})} + (1-\lambda)\sqrt{Ric_S(u,\overline{u})Ric_S(v,\overline{v})}|^2 \\
&\leq (1+tK_1)(\lambda Ric_R(u,\overline{u}) + (1-\lambda)Ric_S(u,\overline{u}))(\lambda Ric_R(v,\overline{v}) + (1-\lambda)Ric_S(v,\overline{v})) \\
&= (1+tK_1)Ric_T(u,\overline{u})Ric_T(v,\overline{v}).
\end{aligned}
$$

Therefore, $V_t$ is convex.

Now let us check that when $t = 0$, the curvature tensor $R_0$ of $(M^n, g)$ is in $V_0$. If we choose $K_2$ very large, then (1) and (3) hold. To check (2), we notice that for fixed $x$, $R_{x\bar{x}p\bar{q}}$ is a Hermitian form. Let $e_i$ be the eigenvectors where $i = 1, 2, ..n$ and

$$R_{x\bar{x}e_i\overline{e_j}} = \delta_{ij}\lambda_i$$

where $\lambda_i$ are all nonpositive. Suppose $u = \sum_{i=1}^{n} u_i e_i, v = \sum_{i=1}^{n} v_i e_i$, then

$$
\begin{aligned}
|R_{x\bar{x}u\bar{v}}|^2 &= |\sum_{i=1}^{n} u_i\overline{v_i}\lambda_i|^2 \\
&\leq (\sum_{i=1}^{n} |u_i\sqrt{-\lambda_i}|^2)(\sum_{i=1}^{n} |\overline{v_i}\sqrt{-\lambda_i}|^2) \qquad (2)\\
&= R_{x\bar{x}u\bar{u}}R_{x\bar{x}v\bar{v}} \\
&\leq Ric(u,\overline{u})Ric(v,\overline{v}).
\end{aligned}
$$

Let us state Hamilton's maximum principle for tensors. Let $M^n$ be a closed oriented manifold with a smooth family of Riemannian metric $g(t)$, $t \in [0, T]$. Let $V \to M$ be a real vector bundle with a time dependent metric $h$ and $\Gamma(V)$ be the vector space of smooth sections on $V$. Let $\nabla_t^L$ denote the corresponding Levi-Civita connection on $(M, g(t))$. Furthermore, let $\nabla_t$ denote a time dependent metric connection on $V$. For a section $R \in \Gamma(V)$, define a new section $\Delta_t R \in \Gamma(V)$ as follows. For $p \in M$ choose an orthonormal basis of $V_p$(the fiber of $V$ at $p$) and extend it along the radial geodesics in $(M, g(t))$ emanating from $p$ by parallel transport of $\nabla_t$ to an orthonormal basis $X_1(q)$, ..., $X_d(q)$ of $V_q$ for all $q$ in a small neighborhood of $p$. If $f_i$ satisfies $R = \sum_{i=1}^{d} f_i X_i$, then

$$(\Delta_t R)(p) = \sum_{i=1}^{d} (\Delta_t f_i)X_i(p)$$

where $\Delta_t$ is the Beltrami Laplacian on functions.

Suppose that a time dependent section $R(\cdot, t) \in \Gamma(V)$ satisfies the parabolic equation

$$\frac{\partial R(p,t)}{\partial t} = (\Delta_t R)(p,t) + f(R(p,t)) \qquad (3)$$

where $f : V \to V$ is a local Lipschitz map mapping each fibre $V_q$ to itself. Roughly speaking, Hamilton's maximum principle says that the dynamics of the parabolic equation (3) is controlled by the ordinary differential equation

$$\frac{dR}{dt} = f(R(p,t)). \tag{4}$$

More precisely, we have the following version of Hamilton's maximum principle in [1] and [2]:

**Theorem 2.** *For $t \in [0,\delta]$, let $C(t) \subseteq V$ be a closed subset, depending continuously on $t$. Suppose that each of the sets $C(t)$ is invariant under parallel transport, fiber-wise convex and that the family of $C(t)(0 \le t \le \delta)$ is invariant under the ordinary differential equation (4). Then for any solution $R(p,t) \in \Gamma(V)$ on $M \times [0,\delta]$ of parabolic equation (3) with $R(\cdot,0) \in C(0)$, we have $R(\cdot,t) \in C(t)$ for all $t \in [0,\delta]$.*

Let us go back to the proof of proposition 1. In view of theorem 2, we just need to prove that $V(t)$ is invariant under the ODE equation of the curvature operator, i.e, we drop the Laplacian in the evolution equation of the curvature operator. For any $R(0) \in V_0$, we consider perturbation $R_\lambda(0) = R(0) - \lambda R'$ for the initial condition of the ODE, where $\lambda$ is a small positive number and $R'$ is the curvature tensor with holomorphic sectional curvature 1. For simplicity, when $\lambda$ is fixed, we use $R$ to denote the solution to the ODE with initial condition $R_\lambda(0)$.

**Lemma.** *There exist positive constants $\epsilon, A, K_1, K_2, K_3$ which are independent of $\lambda$ such that $\epsilon K_1 \le 1$ and for any $t \in [0,\epsilon]$, the solution $R$ satisfies*
*(1'). $Ric(\alpha,\overline{\alpha}) \le -\frac{\lambda}{2}e^{-At}$ for any $e_\alpha \in T^{1,0}M$ and $|e_\alpha|_{g(t)} = 1$.*
*(2'). $|R_{x\overline{x}u\overline{v}}|^2 \le (1+tK_1)Ric(u,\overline{u})Ric(v,\overline{v})$ for any $x,u,v \in T^{1,0}M$ and $|x|_{g(t)} = 1$.*
*(3'). $\|R\| \le K_2 + tK_3$.*

*Proof.* We can find $B > 0$ such that $\|R\| \le B$ for all small $t$ and $\lambda$. Take $K_2 = B$. If $K_3$ is big enough, then (3') will be preserved for small $t$ and $\lambda$.

**Claim 1.** *If $R$ satisfies (1'), (2') and (3') of the Lemma at time $t$, then there exists $C > 0$ depending only on the bound of the curvature tensor such that $|R_{i\overline{j}k\overline{l}}| \le C\sqrt{-Ric(i,\overline{i})}$ and $|R_{i\overline{j}k\overline{l}}| \le C\sqrt{Ric(i,\overline{i})Ric(j,\overline{j})}$ at time $t$ for any $e_i, e_j, e_k, e_j \in T^{1,0}M$ and that the length is 1 in $g(t)$.*

*Proof.* The proof follows if we polarize the curvature tensor. □

In the following, $C$ will denote a positive constant which depends only on the bound of the curvature tensor. $R$ satisfies the ODE

$$\frac{d}{dt}R_{i\overline{j}k\overline{l}} = \sum R_{i\overline{j}**}R_{****} + \sum R_{i***}R_{*\overline{j}**}$$

where $*$ are indices. By Claim 1, we have

$$\left|\frac{d}{dt}R_{i\overline{j}k\overline{l}}\right| \le C\sqrt{Ric(i,\overline{i})Ric(j,\overline{j})}$$

It is easy to see that (1'), (2') and (3') in the Lemma hold for $t = 0$. If the Lemma is not true, let $t_0$ be the first time so that the Lemma fails. There are two possibilities:

(i) (1') does not hold in $[0, t_1)$ for any $t_1 > t_0$.

(ii) (2') does not hold in $[0, t_1)$ for any $t_1 > t_0$.

In case (i), after a slight computation, Claim 1 implies

$$\frac{d}{dt}\left(\frac{Ric(\alpha, \overline{\alpha})}{g(t)(\alpha, \overline{\alpha})}\right) \leq -CRic(\alpha, \overline{\alpha})$$

for $|\alpha|_{g(t)} = 1$. If $A > 2C$, then this is a contradiction.

For case (ii), Claim 1 gives

$$\frac{d}{dt}\left((1 + tK_1)Ric(u, \overline{u})Ric(v, \overline{v}) - \frac{|R_{x\overline{x}u\overline{v}}|^2}{g(t)(x, \overline{x})}\right) \geq (K_1 - C)Ric(u, \overline{u})Ric(v, \overline{v}) > 0 \tag{5}$$

if $|x|_{g(t)} = 1, K_1 > 2C + 10, t_0 < \epsilon < \frac{1}{2K_1}$. Therefore, this is again a contradiction. The Lemma is thus proved. □

Proposition 1 follows if we let $\lambda \to 0$ in the Lemma. □

By theorem 2, $Ric(g(t)) \leq 0$ for small $t > 0$. If $Ric < 0$ for some small $t > 0$, then $c_1(M) < 0$. Otherwise, the rank of the Ricci curvature is less than $n$ for some $t > 0$. We shall show that the rank of $Ric_t$ is constant and the null space is parallel.

We use the arguments in [1](page 676-677). Consider

$$\frac{\partial Ric(v, \overline{v})}{\partial t} = \Delta_t Ric_{v\overline{v}} + \sum R_{v\overline{v}**}R_{****} + \sum R_{v****}R_{*\overline{v}**}.$$

Define $\tilde{Ric}_t = e^{Ht}Ric_t$. By Proposition 1, if $H$ is large, then

$$\frac{\partial \tilde{Ric}_{v\overline{v}}}{\partial t} \leq \Delta_t \tilde{Ric}_{v\overline{v}}. \tag{6}$$

Now we show that the rank of $\tilde{Ric}$ is constant for small $t > 0$. Follow [1], let $0 \geq \mu_1 \geq \mu_2 \geq .... \geq \mu_n$ denote the eigenvalues of $\tilde{Ric}$ and let

$$\sigma_l = \mu_1 + \mu_2 + .... + \mu_l.$$

Fix $p \in M$ and let $e_1(t_0), e_2(t_0), ..., e_l(t_0)$ be an orthogonal basis of $T_p^{1,0}(M)$ such that $\sigma_l(t_0) = \sum_{i=1}^{l} \tilde{Ric}_{t_0}(e_i(t_0), \overline{e_i(t_0)})$. Now

$$\begin{aligned}
\sigma_l'(t_0) : &= \limsup_{t \nearrow t_0} \frac{\sigma_l(t_0) - \sigma_l(t)}{t_0 - t} \\
&\leq \frac{d}{dt}|_{t=t_0} \sum_{i=1}^{l} \tilde{Ric}_t(e_i(t_0), \overline{e_i(t_0)}) \\
&\leq \sum_{i=1}^{l} \Delta \tilde{Ric}_{t_0}(e_i(t_0), \overline{e_i(t_0)}) \\
&\leq \Delta\sigma_l
\end{aligned} \tag{7}$$

Thus

$$\frac{\partial \sigma_l}{\partial t} \leq \Delta\sigma_l$$

in the support function sense. By the strong maximum principle, either $\sigma_l < 0$ for all small $t > 0$ or $\sigma_l \equiv 0$. This proves that $\tilde{Ric}$ has constant rank for small $t > 0$.

Let $v(t) \in T^{1,0}M$ be a smooth vector field on $M$ depending smoothly on $t$ such that $\tilde{Ric}_t(v, \overline{v}) = 0$. Since $\tilde{Ric} \leq 0$, from (6),

$$0 = (\frac{\partial}{\partial t}\tilde{Ric})(v, \overline{v}) \leq \sum_{i=1}^{n} \tilde{Ric}(\nabla_{e_i} v, \overline{\nabla_{e_i} v})$$

where $e_i \in T^{1,0}M$ is a local unitary frame on $M$. This shows that the rank of $Ric_t$ is constant and the null space of $Ric_t$ is parallel. Therefore, $(M, g(t))$ splits locally for all small $t > 0$. Therefore, for metric $g(0)$, the universal cover $\tilde{M}$ is biholomorphic and isometric to $\mathbb{C}^k \times Y^{n-k}$ with the product metric. Note that the Ricci flow on $M$ preserves the local product structure, and for $\epsilon > t > 0$, the Ricci curvature on $Y$ is strictly negative.

The rest proof of Theorem 1 uses the argument of Wu and Zheng [11]. For reader's convenience, we recall some details here. Denote by $\Gamma$ the deck transformation group. For each $0 \leq t < \epsilon$, denote by $I_1, I_2(t)$ the group of holomorphic isometries of $\mathbb{C}^k$ and $Y^{n-k}$ at time $t$. Any $f \in \Gamma$ induces a biholomorphism and isometry on $\mathbb{C}^k \times Y^{n-k}$ for any $0 \leq t < \epsilon$. Therefore $f = (f_1, f_2)$, where $f_1 \in I_1$, $f_2 \in \cap_{0\leq t<\epsilon} I_2(t)$. Denote by $p_i : \Gamma \to I_i$ the projection map, and by $\Gamma_i = p_i(\Gamma)$ the image groups for $i = 1, 2$. Below are two key claims in [11]:

**Claim 2.** *The group $\Gamma_2$ is discrete.*

**Claim 3.** *There exists a finite index subgroup of $\Gamma' \subseteq \Gamma$ such that $\Gamma'_2$ acts freely on $Y$, and $\Gamma'_1$ contains translation only. Here $\Gamma'_i = p_i(\Gamma'), i = 1, 2$.*

Wu and Zheng proved the two claims by using ideas in Eberlein [3][4] and Nadel [10]. For our case, Claim 2 follows by applying Wu and Zheng's argument to $g(t)$ for small $t > 0$(note that in this case $Ric(Y) < 0$). For Claim 3, Wu and Zheng's proof can be carried out without any modification.

By Claim 2 and Claim 3, we have a finite covering $M' = \tilde{M}/\Gamma'$ over $M$, and a holomorphic surjection $q : M' \to N$ induced by the projection from $\tilde{M}$ to $Y$. Here $N = Y/\Gamma'_2$ is a compact Kähler manifold. $q$ makes $M'$ a holomorphic fibre bundle over $N$ with fibre being complex torus. $M'$ is also isometric to a flat torus bundle over $N$. This completes the proof of Theorem 1. □

**Remark.** *The analogous result of Proposition 1 is true for the Riemannian case, i.e, if a compact manifold has nonpositive sectional curvature, then after the Ricci flow, in a short time, the Ricci curvature will be nonpositive.*

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